Question 1

The stemplot in Figure 1 shows the distribution of the average age, in years, at which women first marry in 17 countries.

average age, in years, of women at first marriage

key: 27|3 represents 27.3 years

24	
25	0
26	6
27	1 1 3 4 7
28	2 2 2 3 3 6
29	1 1
30	1 4
31	

Median

n=17

$$\frac{n+1}{2}=\frac{(17+1)}{2}=9 \quad median\ is\ the\ 9th\ value$$

The 9^th^ value is 28.2

Figure 1

a. For these countries, determine

i. the lowest average age of women at first marriage

25.0 years

ii. the median average age of women at first marriage.

Easier to calculate by hand.

28.2 years

1 + 1 = 2 marks

The stemplot in Figure 2 shows the distribution of the average age, in years, at which men first marry in 17 countries.

average age, in years, of men at first marriage

key: 32|5 represents 32.5 years

25	
26	[0
27	
28	9
29	0 9 9
30	0 0 3] 5 [6 7 9
31	0 0 2
32	5 9]
33	

n=17

$$\frac{n+1}{2}=\frac{(17+1)}{2}=9 \quad median\ is\ the\ 9th\ value$$

The 9^th^ value is 30.5 so M = 30.5

Q_1 is the median of the lower half and Q_3 is the median of the upper half.

$$Q_1=\frac{(29.9+29.9)}{2}=29.9 \qquad Q_3=\frac{(31.0+31.0)}{2}=31.0$$

Figure 2

b. For these countries, determine the interquartile range (IQR) for the average age of men at first marriage.

IQR = Q_3 - Q_1

= 31-29.9

= 1.1 years

1 mark

Core – Question 1 - continued

c. If the data values displayed in Figure 2 were used to construct a boxplot with outliers, then the country for which the average age of men at first marriage is 26.0 years would be shown as an outlier.
Explain why this is so. Show an appropriate calculation to support your explanation.

Lower fence = $Q_1 - 1.5 \times IQR$
= $29.9 - 1.5 \times 1.1$
= 28.25

Since the value of 26 is less than the value of the lower fence, 26 is an outlier.

2 marks

Question 2

Table 1 shows information about a particular country. It shows the percentage of women, by age at first marriage, for the years 1986, 1996 and 2006.

Table 1

Age of women at first marriage	Year of marriage		
	1986	1996	2006
19 years and under	8.5%	3.7%	2.0%
20 to 24 years	42.1%	31.3%	21.5%
25 to 29 years	23.4%	31.7%	34.5%
30 years and over	26.0%	33.3%	42.0%

a. Of the women who first married in 1986, what percentage were aged 20 to 29 years inclusive?

31.3 + 31.7 = 63

1 mark

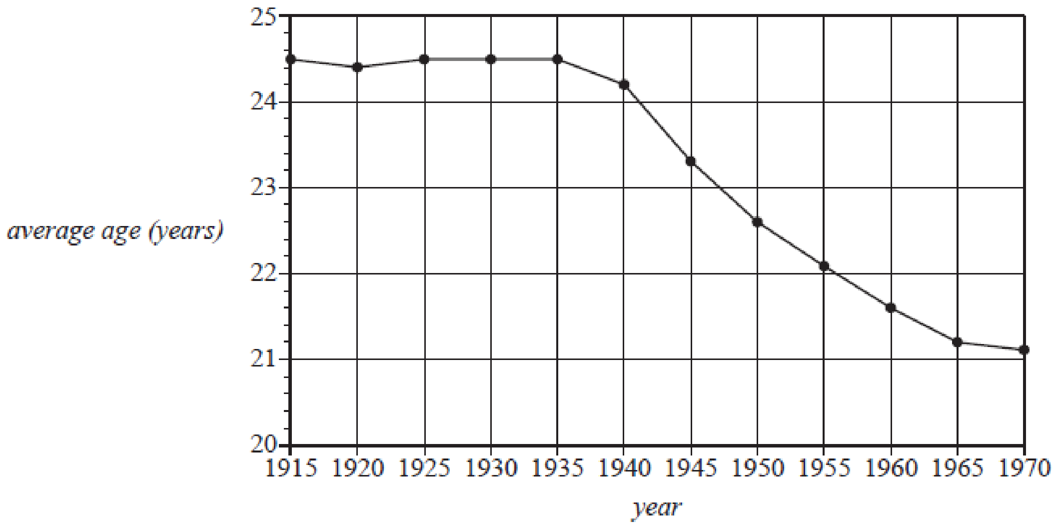
b. Does the information in Table 1 support the opinion that, for the years 1986, 1996 and 2006, the age of women at first marriage was associated with year of marriage?
Justify your answer by quoting appropriate percentages. It is sufficient to consider one age group only when justifying your answer.

There is an association with women marrying at an older age tending to increase from 1986 to 1996 and from 1996 to 2006. The percentage of women at first marriage for 30 years and over increased from 26% (1986) to 33.3% (1996) to 42% (2006).

2 marks

Question 3

The following time series plot shows the average age of women at first marriage in a particular country during the period 1915 to 1970.



a. Use this plot to describe, in general terms, the way in which the average age of women at first marriage in this country has changed during the period 1915 to 1970.

The average age of women at first marriage was relatively stable at approximately 24.5 years between 1915 and 1935. Then the ages decreased from 1935 (24.5 years) to 1970 (21.1 years).

1 mark

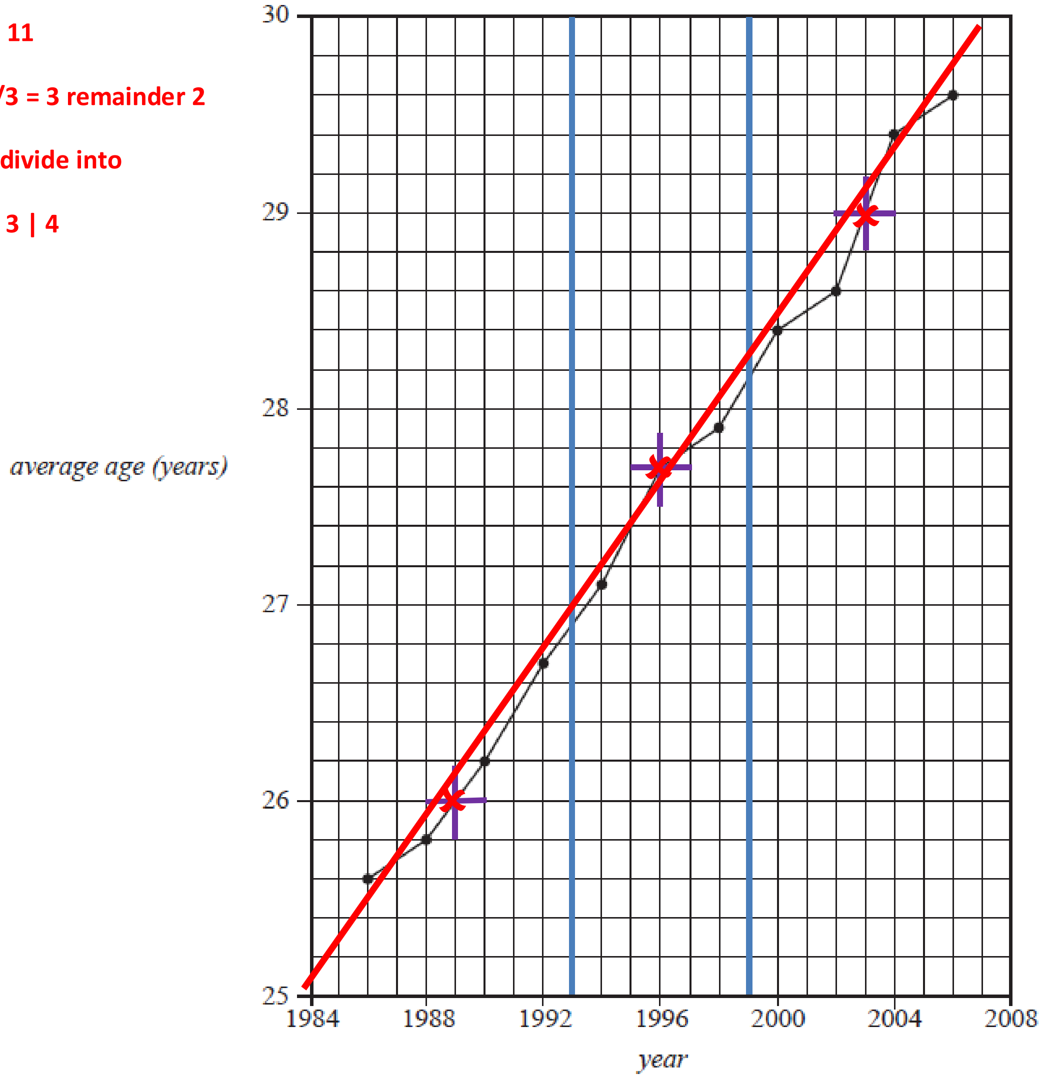
During the period 1986 to 2006, the average age of men at first marriage in a particular country indicated an increasing linear trend, as shown in the time series plot below.

n = 11

11/3 = 3 remainder 2

So divide into

4 | 3 | 4



A three-median line could be used to model this trend.

b. On the graph above

i. clearly mark with a cross (×) the three points that would be used to fit a three-median line to this time series plot

ii. draw in the three-median line.

2 + 1 = 3 marks

The line should be parallel to a line passing through the left and right summary points and moved 1/3 towards the middle summary point.

Question 4

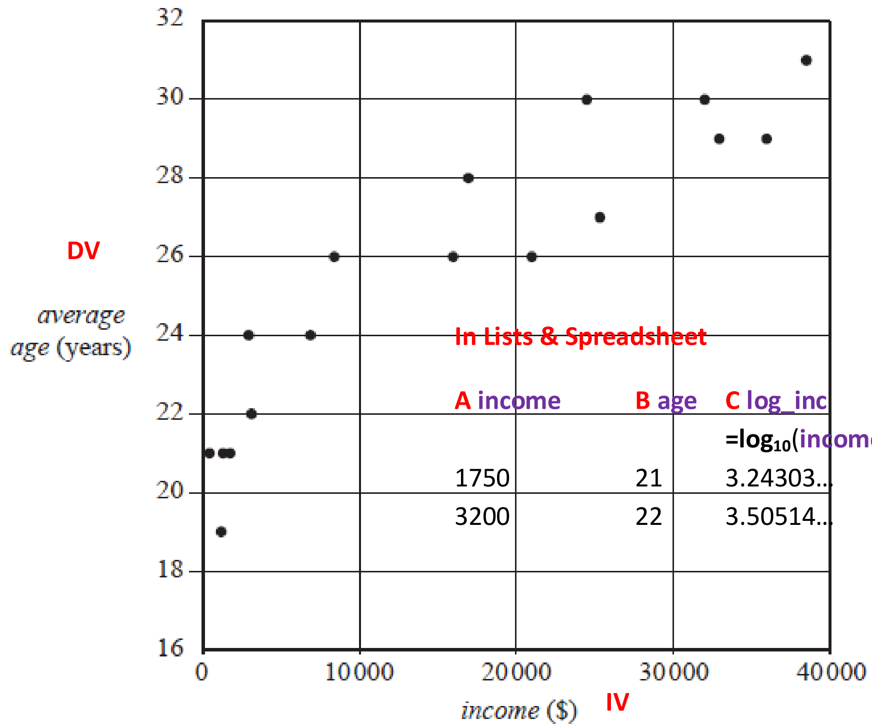
The average age of women at first marriage in years (*average age*) and average yearly income in dollars per person (*income*) were recorded for a group of 17 countries.

The results are displayed in Table 2. A scatterplot of the data is also shown.

Table 2

DV *average age* (years)	IV *income* ($)
21	1750
22	3200
26	8600
26	16000
28	17000
26	21000
30	24500
30	32000
31	38500
29	33000
27	25500
29	36000
19	1300
21	600
24	3050
24	6900
21	1400

From the scatterplot average age is the dependent variable and income is the independent variable.



In Lists & Spreadsheet

A income	**B age**	**C log_inc**
		=log$_{10}$(income)
1750	21	3.24303..
3200	22	3.50514..

In calculator window MENU 6 1 4

XLIST log_inc **New IV**

YLIST age **DV**

The relationship between *average age* and *income* is nonlinear.

A **log transformation** can be applied to the variable *income* and used to linearise the scatterplot.

a. Apply this log transformation to the data and determine the equation of the least squares regression line that allows *average age* to be predicted from log (*income*).

Write the coefficients for this equation, correct to two decimal places, in the spaces provided.

average age = **2.39** + **5.89** × log (*income*)

2 marks

b. Use this equation to predict the average age of women at first marriage in a country with an average yearly income of \$20000 per person.

Write your answer correct to one decimal place.

$$average\ \ age = 2.39 + 5.89 \times \log(income)$$
$$= 2.39 + 5.89 \times \log(20000)$$
$$= 27.7$$

1 mark

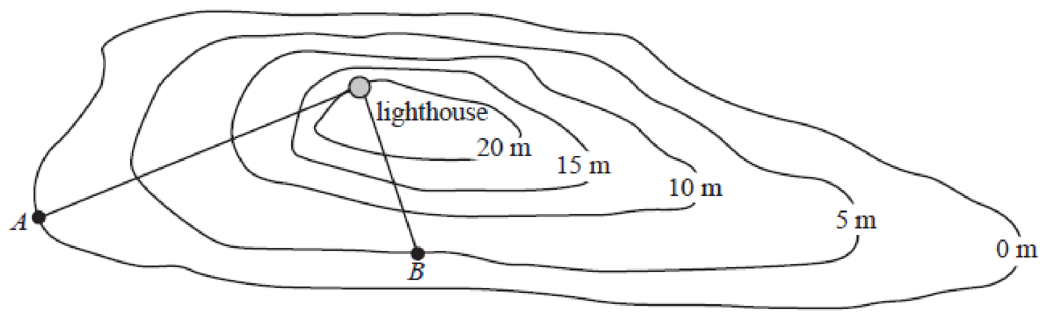
Total 15 marks

End of Core

Question 1

A lighthouse is located on a hill overlooking the sea. A contour map of the hill is shown below. The lighthouse is located at an altitude of 20 metres.

Two points, *A* and *B*, are shown on the contour map.



On the contour map, 1 centimetre represents 30 metres on the **horizontal** level.
On the contour map, the length of the line from point *A* to the lighthouse is 5 centimetres.

a. Determine the horizontal distance, in metres, from point *A* to the lighthouse.

1 cm represents 30 metres

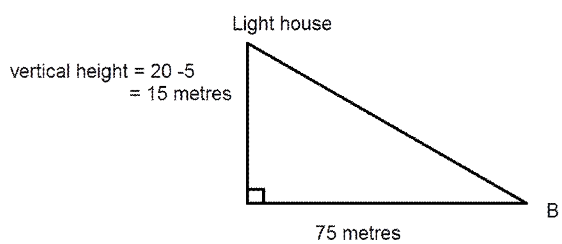
5 cm represents 150 metres

1 mark

The horizontal distance from point *B* to the lighthouse is 75 metres.

b. Calculate the average slope between point *B* and the lighthouse.

1 mark



$$m = \frac{rise}{run}$$
$$= \frac{15}{75}$$
$$= 0.2$$

Question 2

Ship A and Ship B can both be seen from the lighthouse.
Ship A is 5 kilometres from the lighthouse, on a bearing of 028°.
Ship B is 5 kilometres from Ship A, on a bearing of 130°.
This information is shown in Figure 1 below.

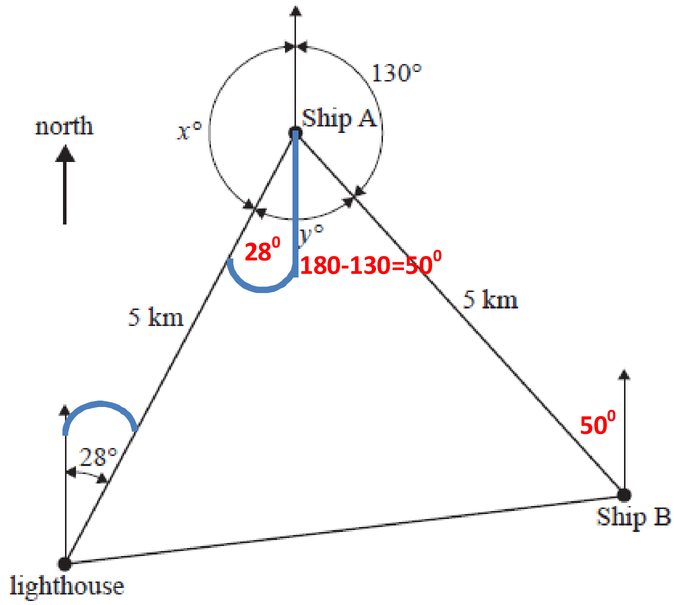


Figure 1

a. Two angles, x and y, are shown in Figure 1 above.

i. Determine the size of the angle x in degrees.

x = 360 – 50 – 28 = 152⁰

ii. Determine the size of the angle y in degrees.

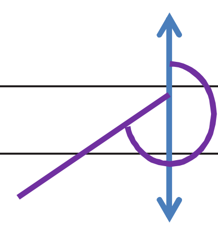
y= 28 + 55 = 78⁰

1 + 1 = 2 marks

b. Determine the bearing of the lighthouse from Ship A.

Bearing = 130 + 50 + 28

= 208⁰



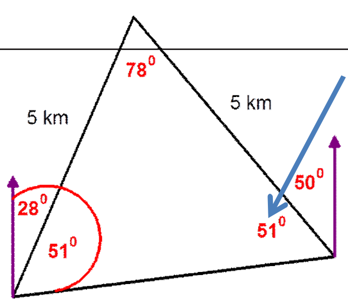
Bearing is the angle clockwise from North

1 mark

c. Determine the bearing of Ship B from the lighthouse.

Bearing = 079⁰

Note: 3 digits were required for the bearing.



Isosceles triangle with two equal angles.

1 mark

Question 3

The lighthouse has a lightroom, shown shaded in Figure 2 below.
The floor of the lightroom is in the shape of a regular octagon.
The longest distance across the floor is 4 metres.

The lightroom floor and $\angle POQ = \theta°$ are shown in Figure 3 below.

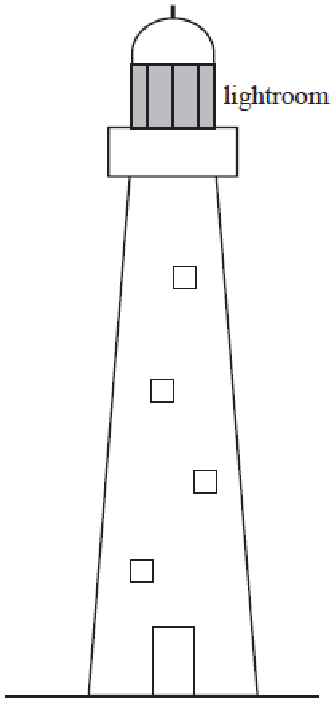


Figure 2

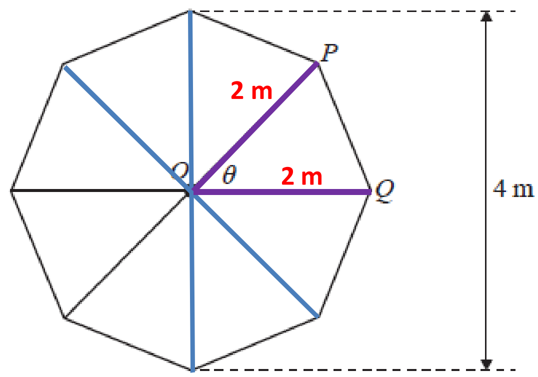


Figure 3

a. Show that the size of the angle θ is 45°.

$360 \div 8 = 45^0$

1 mark

b. Determine the area of triangle POQ.
Write your answer in square metres correct to one decimal place.

$A = ½ \times 2 \times 2 \times \sin(45)$

$=1.4$

1 mark

The lightroom is surrounded by a walkway of diameter 6.4 metres.
An outer circular wall surrounds the walkway.
The walkway is shown shaded in Figure 4 below.

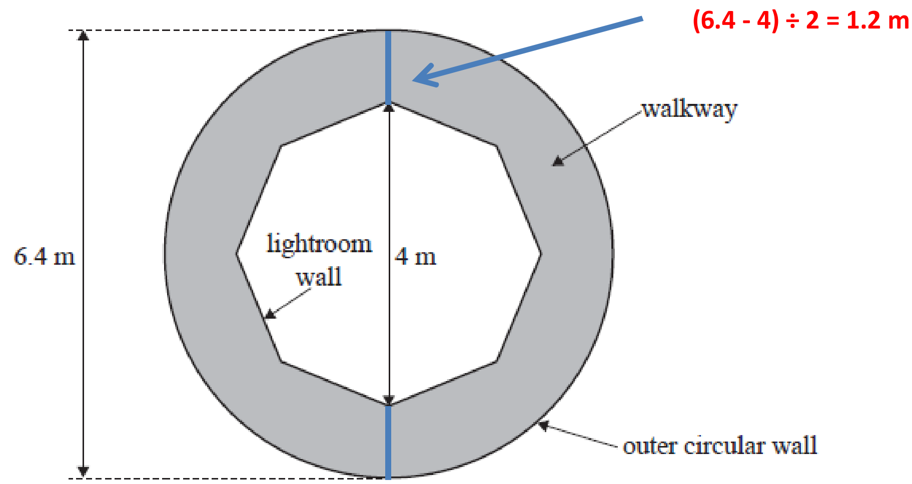


Figure 4

c. Determine the minimum distance between the lightroom wall and the outer circular wall.

1.2 metres

1 mark

d. The walkway is the shaded area in Figure 4. Determine its area correct to the nearest square metre.

$$A_{shaded\ area} = A_{circle} - A_{octagon}$$

$$= \pi \times 3.2^2 - \frac{1}{2} \times 2 \times 2 \times \sin(45) \times 8$$

$$= 21$$

2 marks

Question 4

The lighthouse tower, shaded on Figure 5 below, is in the shape of a truncated cone.
It has circular cross-sections that decrease uniformly from a radius of 3.5 metres at ground level to a radius of 2 metres at the walkway.
The height of the lighthouse tower is 18 metres.
The angle marked α is the angle that the outer wall of the lighthouse tower makes with the horizontal at ground level.

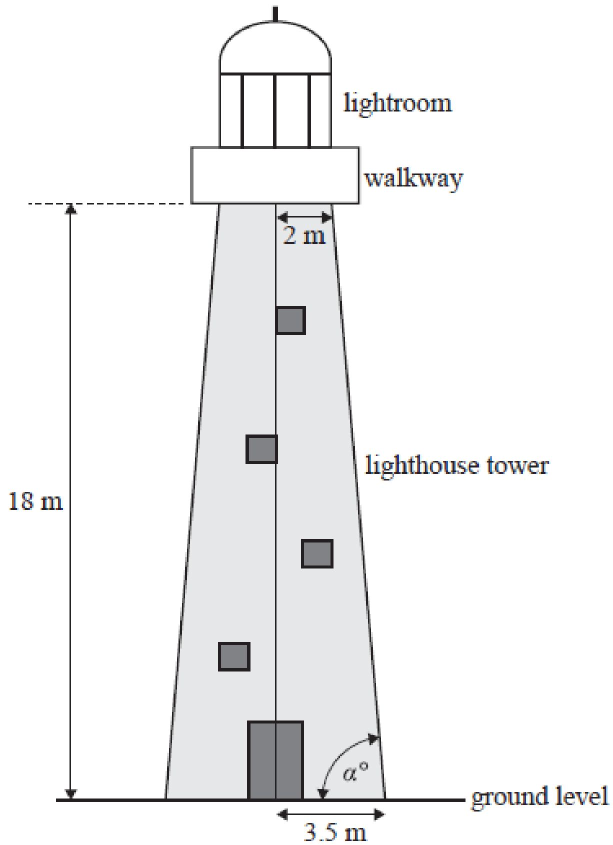
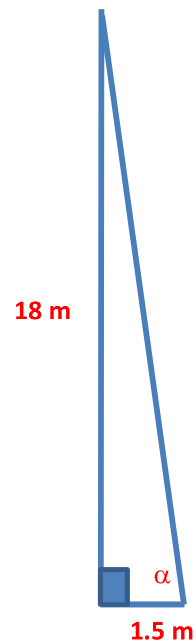


Figure 5



a. Determine the size of the angle α.
Write your answer in degrees correct to one decimal place.

$$\tan(\alpha) = \frac{18}{1.5}$$

$$\alpha = Tan^{-1}\left(\frac{18}{1.5}\right)$$

$$= 85.2$$

1 mark

The lighthouse tower is part of a cone. The height of this cone is h metres and the base radius is 3.5 metres, as shown in Figure 6.

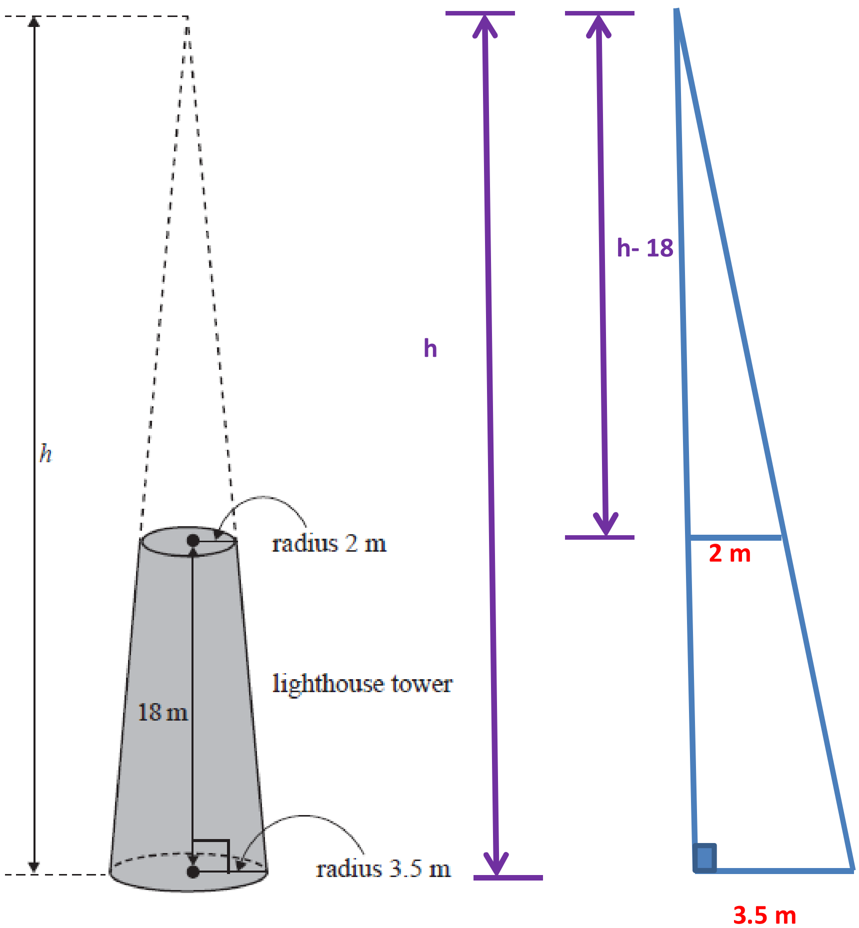


Figure 6

b. **i.** Determine h, the height of this cone, in metres.

h = 42 metres

$$\frac{h}{h-18}=\frac{3.5}{2}$$

$$Solve\left(\frac{h}{h-18}=\frac{3.5}{2},h\right)$$

Don't write this calculator step in the exam.

ii. Determine the volume of the lighthouse tower.
Write your answer to the nearest cubic metre.

$$V_{tower}=V_{\text{large cone}}-V_{\text{small cone}}$$

$$=\frac{1}{3}\times\pi\times3.5^2\times42-\frac{1}{3}\times\pi\times2^2\times(42-18)$$

$$=438\ m^3$$

2 + 1= 3 marks

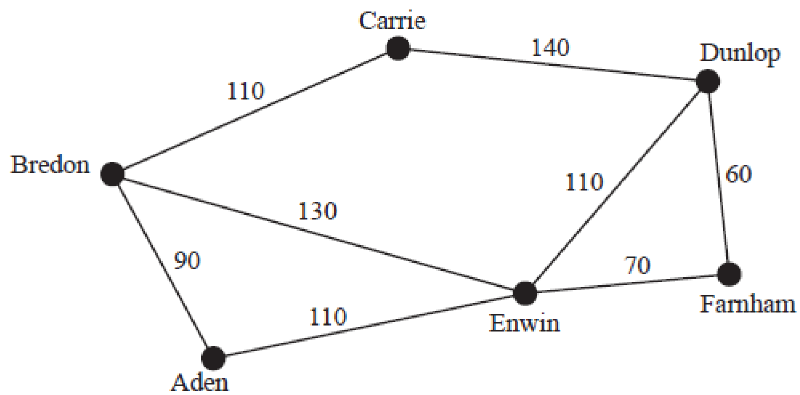
Total 15 marks

END OF MODULE 2

Module 5: Networks and decision mathematics

Question 1

Aden, Bredon, Carrie, Dunlop, Enwin and Farnham are six towns.
The network shows the road connections and distances between these towns in kilometres.



a. In kilometres, what is the shortest distance between Farnham and Carrie?

Farnham→ Dunlop→Carrie 60 + 140 = 200 km

1 mark

b. How many different ways are there to travel from Farnham to Carrie without passing through any town more than once?

6 ways: FDC, FDEABC, FDEBC, FEDC, FEABC, FEBC

1 mark

An engineer plans to inspect all of the roads in this network.
He will start at Dunlop and inspect each road only once.

Euler line, so need to start and finsh at an odd degree vertex.

c. At which town will the inspection finish?

Bredon (degree = 3)

1 mark

Another engineer decides to start and finish her road inspection at Dunlop.
If an assistant inspects two of the roads, this engineer can inspect the remaining six roads and visit each of the other five towns only once.

Uses 6 roads and visits all towns once. Hamilton circuit for the engineer.

d. How many kilometres of road will the assistant need to inspect?

Engineer travels a Hamilton circuit – going around the outside.

Distance travelled by assistant.

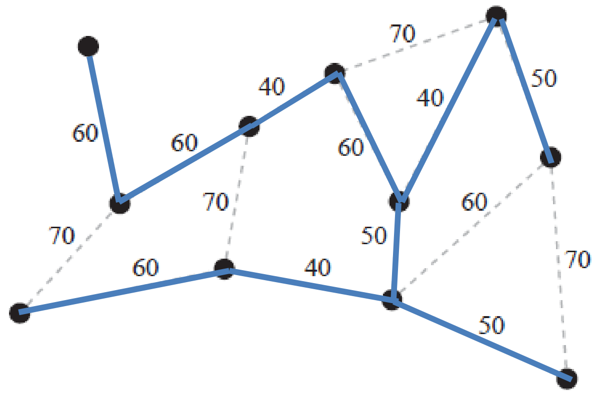
1 mark

Assistance = 130 + 110

= 240 km

Question 2

At the Farnham showgrounds, eleven locations require access to water. These locations are represented by vertices on the network diagram shown below. The dashed lines on the network diagram represent possible water pipe connections between adjacent locations. The numbers on the dashed lines show the minimum length of pipe required to connect these locations in metres.



Minimum spanning tree so need to use Prim's algorithm.

All locations are to be connected using the smallest total length of water pipe possible.

a. On the diagram, show where these water pipes will be placed.

1 mark

b. Calculate the total length, in metres, of water pipe that is required.

60 + 60 + 40 + 60 + 40 + 50 + 50 + 50 + 40 + 60 = 510 m

1 mark

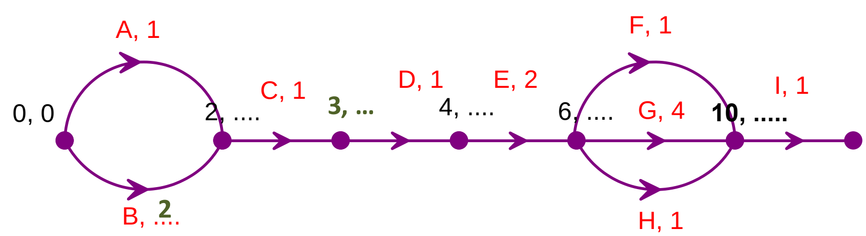
Question 3

A section of the Farnham showgrounds has flooded due to a broken water pipe. The public will be stopped from entering the flooded area until repairs are made and the area has been cleaned up.

The table below shows the nine activities that need to be completed in order to repair the water pipe. Also shown are some of the durations, Earliest Start Times (EST) and the immediate predecessors for the activities.

Activity	Activity description	Duration (hours)	EST	Immediate predecessor(s)
A	Erect barriers to isolate the flooded area	1	0	–
B	Turn off the water to the showgrounds		0	–
C	Pump water from the flooded area	1	2	A, B
D	Dig a hole to find the broken water pipe	1		C
E	Replace the broken water pipe	2	4	D
F	Fill in the hole	1	6	E
G	Clean up the entire affected area	4	6	E
H	Turn on the water to the showgrounds	1	6	E
I	Take down the barriers	1	10	F, G, H

Easier to quickly draw an appropriate network with the given information shown.



a. What is the duration of activity B?

2 hours as the earliest starting time for activity C is 2 weeks.

1 mark

b. What is the Earliest Start Time (EST) for activity D?

3 weeks by continuing the forward pass.

1 mark

c. Once the water has been turned off (Activity B), which of the activities C to I could be delayed without affecting the shortest time to complete all activities?

Critical path (longest path) is BCDEGI. Can only delay the non critical activities which are F and H.

1 mark

It is more complicated to replace the broken water pipe (Activity E) than expected. It will now take four hours to complete instead of two hours.

d. Determine the shortest time in which activities A to I can now be completed.

Critical path (longest path) is BCDEGI. Length is 2 + 1 + 1 + 4 + 4 + 1 = 13 weeks.

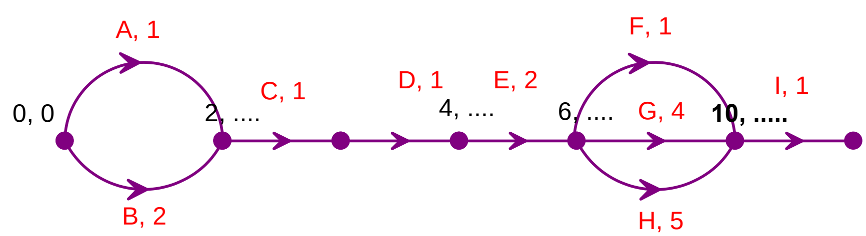
1 mark

Turning on the water to the showgrounds (Activity H) will also take more time than originally expected. It will now take five hours to complete instead of one hour.

e. With the increased duration for Activity H and Activity E, determine the shortest time in which activities A to I can be completed.

Critical path (longest path) is BCDEHI. Length is 2 + 1 + 1 + 4 + 5 + 1 = 14 weeks.

1 mark

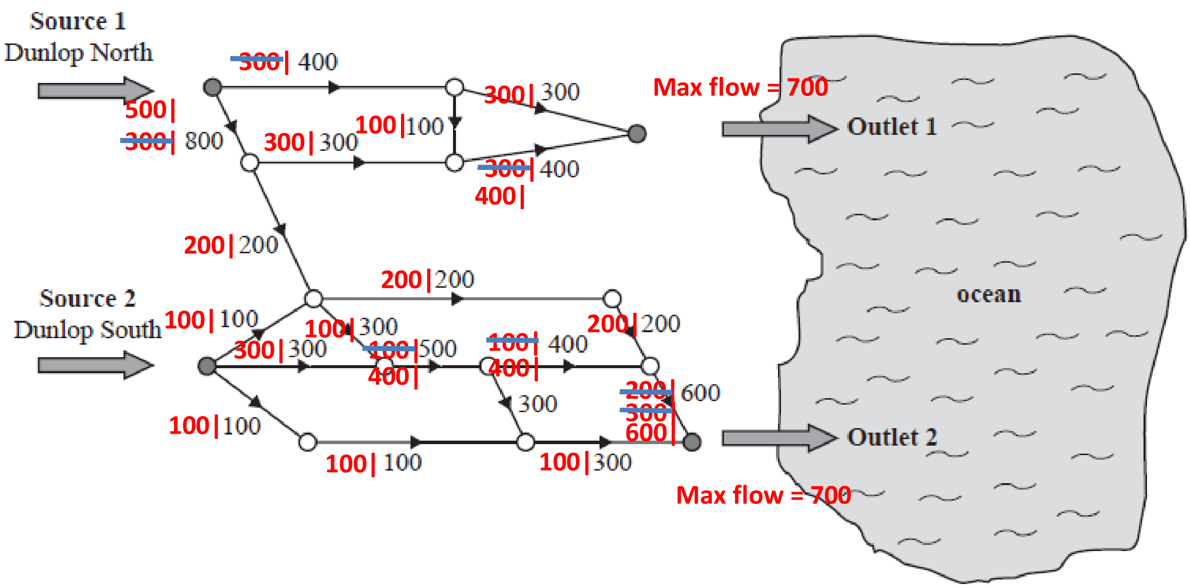


Question 4

Stormwater enters a network of pipes at either Dunlop North (Source 1) or Dunlop South (Source 2) and flows into the ocean at either Outlet 1 or Outlet 2.

On the network diagram below, the pipes are represented by straight lines with arrows that indicate the direction of the flow of water. Water cannot flow through a pipe in the opposite direction.

The numbers next to the arrows represent the maximum rate, in kilolitres per minute, at which stormwater can flow through each pipe.



a. Complete the following sentence for this network of pipes by writing either the number 1 or 2 in each box.

Stormwater from Source $\boxed{2}$ **cannot** reach Outlet $\boxed{1}$

1 mark

b. Determine the maximum rate, in kilolitres per minute, that water can flow from these pipes into the ocean at

Outlet 1

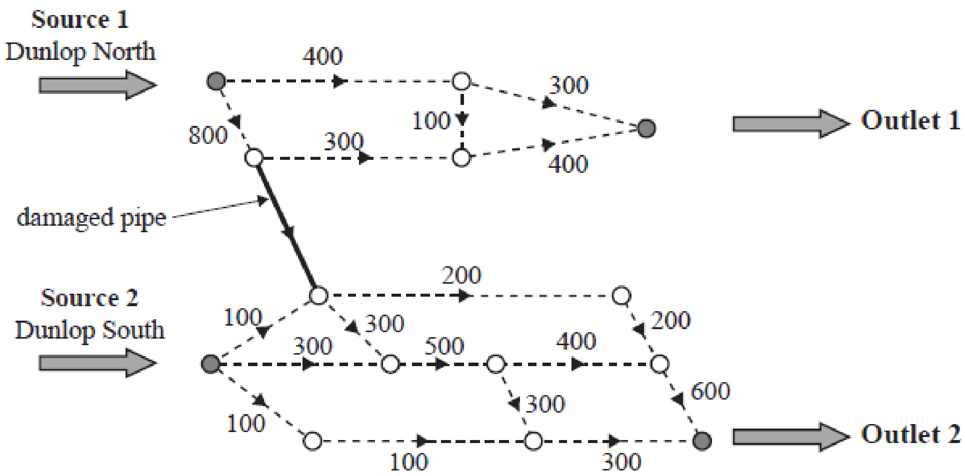
700

Outlet 2

700

2 marks

A length of pipe, shown in **bold** on the network diagram below, has been damaged and will be replaced with a larger pipe.

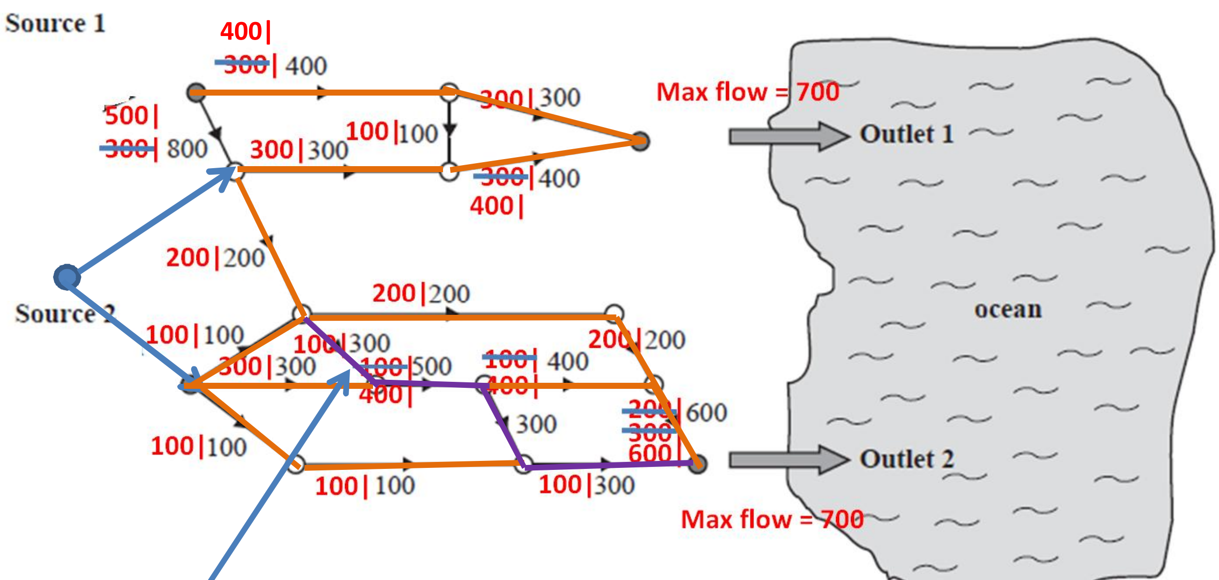


c. The new pipe must enable the greatest possible rate of flow of stormwater into the ocean from Outlet 2.
What minimum rate of flow through the pipe, in kilolitres per minute, will achieve this?

300

1 mark

Total 15 marks

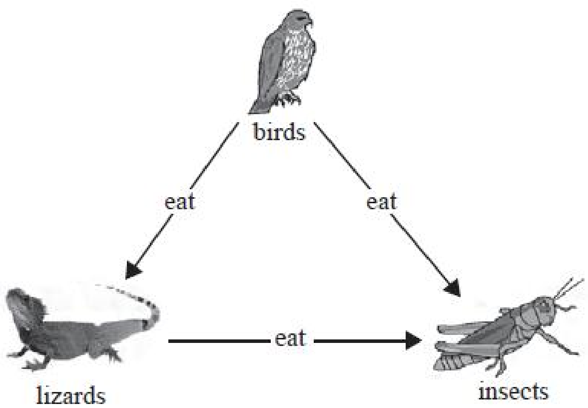


Only pipes with spare capacity and only an extra 100 can be put through these pipes.

Question 1

The diagram below shows the feeding paths for insects (I), birds (B) and lizards (L). The matrix E has been constructed to represent the information in this diagram. In matrix E, a ‘1’ is read as ‘eat’ and a ‘0’ is read as ‘do not eat’.

Read column to row, which is not the usual convention.



$$E = \begin{array}{c} \begin{array}{ccc} I & B & L \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ I \\ B \\ L \end{array}$$

a. Referring to insects, birds or lizards

i. what does the ‘1’ in column B, row L, of matrix E indicate

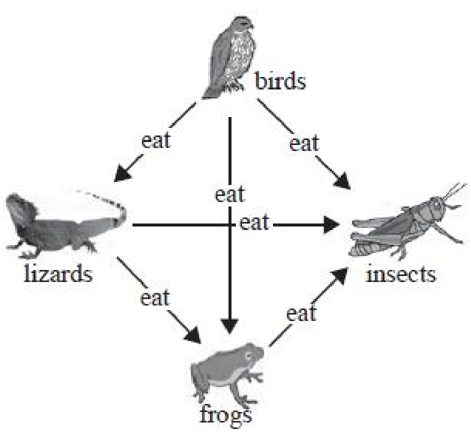
Birds eat lizards.

ii. what does the row of zeros in matrix E indicate?

Neither insects nor birds nor lizards eat birds.

1 + 1 = 2 marks

The diagram below shows the feeding paths for insects (I), birds (B), lizards (L) and frogs (F).
The matrix Z has been set up to represent the information in this diagram.
Matrix Z has not been completed.



$$Z = \begin{array}{c} \begin{array}{cccc} I & B & L & F \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & \underline{1} \\ 0 & 0 & 0 & \underline{0} \\ 0 & 1 & 0 & \underline{0} \\ \underline{0} & \underline{1} & \underline{1} & \underline{0} \end{bmatrix} \end{array} \begin{array}{c} \\ I \\ B \\ L \\ F \end{array}$$

b. Complete the matrix Z above by writing in the seven missing elements.

1 mark

Question 2

To reduce the number of insects in a wetland, the wetland is sprayed with an insecticide.
The numbers of insects (I), birds (B), lizards (L) and frogs (F) in the wetland that has been sprayed with insecticide are displayed in the matrix N below.

$$N = \begin{array}{c} \begin{array}{cccc} I & B & L & F \end{array} \\ \begin{bmatrix} 100\,000 & 400 & 1000 & 800 \end{bmatrix} \end{array}$$

Unfortunately, the insecticide that is used to kill the insects can also kill birds, lizards and frogs.
The proportions of insects, birds, lizards and frogs that have been killed by the insecticide are displayed in the matrix D below.

$$D = \begin{array}{cl} \textit{alive before spraying} & \\ \begin{array}{cccc} I & B & L & F \end{array} & \\ \begin{bmatrix} 0.995 & 0 & 0 & 0 \\ 0 & 0.05 & 0 & 0 \\ 0 & 0 & 0.025 & 0 \\ 0 & 0 & 0 & 0.30 \end{bmatrix} & \begin{array}{l} I \\ B \\ L \\ F \end{array} \textit{dead after spraying} \end{array}$$

Read column to row, which is not the usual convention.

a. Evaluate the matrix product $K = ND$.

$$K = \begin{array}{c} \textit{dead after spraying} \\ \begin{array}{cccc} I & B & L & F \end{array} \\ \begin{bmatrix} 99500 & 20 & 25 & 240 \end{bmatrix} \end{array}$$

1 mark

b. Use the information in matrix K to determine the number of birds that have been killed by the insecticide.

Number killed of birds killed = 20

1 mark

c. Evaluate the matrix product $M = KF$, where $F = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

Need to realise how to multiply by hand.

$M = \begin{bmatrix} 285 \end{bmatrix}$

Birds killed **Lizards killed** **Frogs killed**

$$\begin{bmatrix} 99500 & 20 & 25 & 240 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 99500 \times 0 + 20 \times 1 + 25 \times 1 + 240 \times 1 \end{bmatrix} = \begin{bmatrix} 285 \end{bmatrix}$$

1 mark

d. In the context of the problem, what information does matrix M contain?

Total number of birds, lizards and frogs killed by the insecticide.

1 mark

Question 3

A breeding program is started in the wetlands. It is aimed at establishing a colony of native ducks.

The matrix W_0 displays the number of juvenile female ducks (J) and the number of adult female ducks (A) that are introduced to the wetlands at the start of the breeding program.

$$W_0 = \begin{array}{c} \begin{bmatrix} 32 \\ 64 \end{bmatrix} \end{array} \begin{array}{c} J \\ A \end{array}$$

a. In total, how many female ducks are introduced to the wetlands at the start of the breeding program?

32 + 64 = 96 females

1 mark

The number of juvenile female ducks (J) and the number of adult female ducks (A) in the colony at the end of Year 1 of the breeding program is determined using the matrix equation

$$W_1 = BW_0$$

Not a transition matrix numbers at successive years will either increase or decrease.

In this equation, B is the breeding matrix

$$B = \begin{array}{cc} J & A \\ \end{array}$$
$$B = \begin{bmatrix} 0 & 2 \\ 0.25 & 0.5 \end{bmatrix} \begin{array}{c} J \\ A \end{array}$$

Now 168 female ducks, with a higher proportion of juveniles.

b. Determine W_1

$$W_1 = \begin{bmatrix} 0 & 2 \\ 0.25 & 0.5 \end{bmatrix} \times \begin{bmatrix} 32 \\ 64 \end{bmatrix} = \begin{bmatrix} 128 \\ 40 \end{bmatrix}$$

1 mark

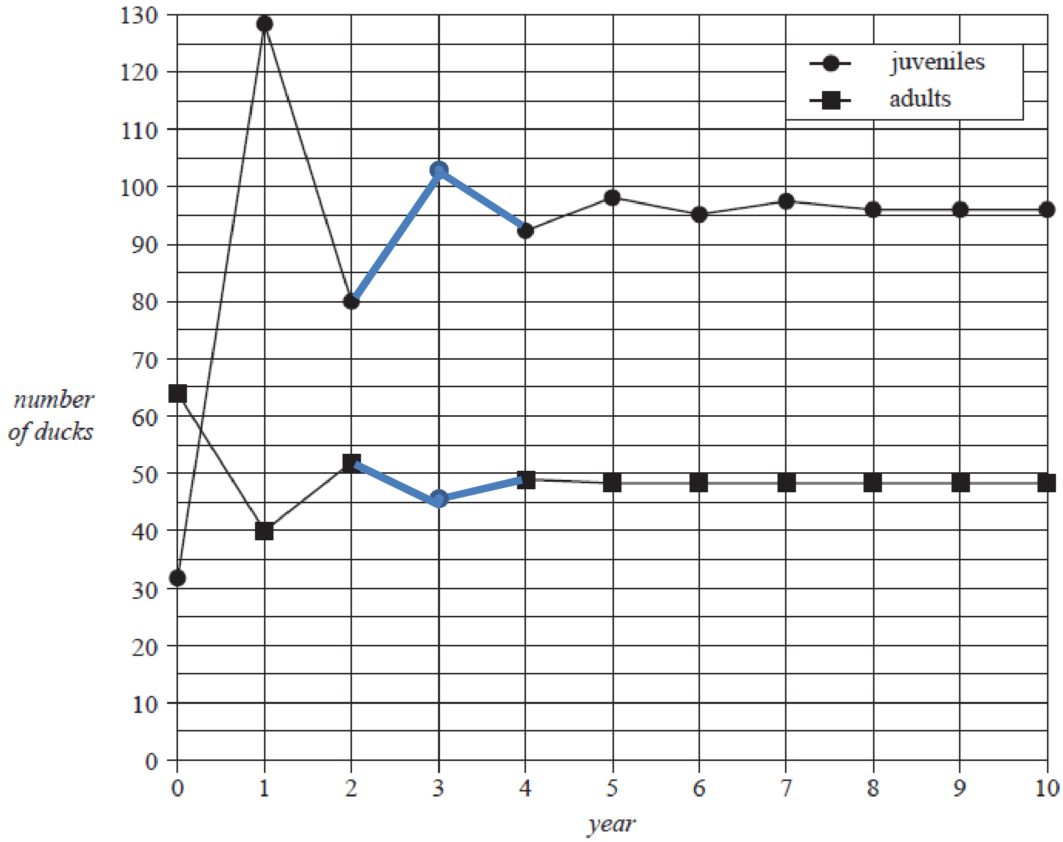
Module 6: Matrices continued

ci

The number of juvenile female ducks (J) and the number of adult female ducks (A) in the colony at the end of Year n of the breeding program is determined using the matrix equation

$$W_n = BW_{n-1}$$

The graph below is incomplete because the points for the end of Year 3 of the breeding program are missing.



c. i. Use matrices to calculate the number of juvenile and the number of adult female ducks expected in the colony at the end of Year 3 of the breeding program.

Plot the corresponding points on the graph.

ii. Use matrices to determine the expected total number of female ducks in the colony in the long term. Write your answer correct to the nearest whole number.

Use calculator to continue iterative process until results stabilise. 96 juveniles and 48 adults.

$$W_{20} = B^{20} \times W_0 = \begin{bmatrix} 96 \\ 48 \end{bmatrix} \qquad W_{21} = B^{21} \times W_0 = \begin{bmatrix} 96 \\ 48 \end{bmatrix}$$

ci. $W_2 = B^2 \times W_0 = \begin{bmatrix} 80 \\ 52 \end{bmatrix} \qquad W_3 = B^3 \times W_0 = \begin{array}{c} \begin{bmatrix} 104 \\ 46 \end{bmatrix} \end{array}\begin{array}{l} J \\ A \end{array}$

The breeding matrix B assumes that, on average, each adult female duck lays and hatches two female eggs for each year of the breeding program.

If each adult female duck lays and hatches only one female egg each year, it is expected that the duck colony in the wetland will not be self-sustaining and will, in the long run, die out.

The matrix equation

$$W_n = PW_{n-1}$$

Not a transition matrix so will have to use ite.

with a different breeding matrix

$$P = \begin{array}{c} \\ \\ \end{array}\begin{array}{cc} J & A \\ \left[\begin{array}{cc} 0 & 1 \\ 0.25 & 0.5 \end{array}\right] \end{array}\begin{array}{c} \\ J \\ A \end{array}$$

Initially 32+64=96 ducks

and the initial state matrix

$$W_0 = \begin{bmatrix} 32 \\ 64 \end{bmatrix}\begin{array}{c} J \\ A \end{array}$$

models this situation.

d. During which year of the breeding program will the number of female ducks in the colony halve?

End of Year 4 $W_4 = P^4 \times W_0 = \begin{bmatrix} 28 \\ 23 \end{bmatrix}$ **End of Year 5** $W_5 = P^5 \times W_0 = \begin{bmatrix} 23 \\ 18.5 \end{bmatrix}$ **so a total of 41.5 ducks.**

Hence during the 5th year the number of ducks is halved.

1 mark

Changing the number of juvenile and adult female ducks at the start of the breeding program will also change the expected size of the colony.

e. Assuming the same breeding matrix, P, determine the number of juvenile ducks and the number of adult ducks that should be introduced into the program at the beginning so that, at the end of Year 2, there are 100 juvenile female ducks and 50 adult female ducks.

$$W_2 = P \times W_1$$

$$W_1 = P^{-1} \times W_2$$ **End of Year 1 100 adults**

$$= \begin{bmatrix} 0 \\ 100 \end{bmatrix}$$

$$W_1 = P \times W_0$$

$$W_0 = P^{-1} \times W_1$$ **Initially 400 juveniles 0 adults**

$$= \begin{bmatrix} 400 \\ 0 \end{bmatrix}$$

Total 15 marks

OR

$$W_0 = P^{-2} \times W_2 \qquad W_2 = \begin{bmatrix} 100 \\ 50 \end{bmatrix}\begin{array}{c} J \\ A \end{array}$$

$$W_0 = \begin{bmatrix} 400 \\ 0 \end{bmatrix}$$