
Further Maths Matrix Summary

Further Maths Matrix Summary

A matrix is a rectangular array of numbers arranged in rows and columns. The numbers in a matrix are called the **elements** of the matrix. The **order** of a matrix is the number of rows and columns in the matrix.

Example 1

$A = \begin{bmatrix} 3 & 5 \\ 0 & 4 \\ -5 & 6 \end{bmatrix}$ is a **3 by 2 or 3 × 2** matrix as it has 3 rows and 2 columns. Matrices are often denoted by capital letters.

Types of Matrices

$C = \begin{bmatrix} 4 \\ 9 \\ 3 \end{bmatrix}$ is a 3 by 1 **column** matrix and $D = [8 \quad -3 \quad 0 \quad 15]$ is a 1 by 4 **row** matrix.

A **null** or **zero** matrix has all elements zero.

$O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is a 2 by 3 **zero** matrix.

Note that the null matrix is the identity matrix for addition and is often denoted by O .

$F = \begin{bmatrix} 4 & -3 & 0 \\ 5 & -5 & 6 \\ 0 & 4 & 1 \end{bmatrix}$ is a 3 by 3 **square** matrix. A **square** matrix has the same number of rows and columns.

A **diagonal** matrix is a square matrix that has non-zero elements in the leading diagonal only.

$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ is a diagonal matrix.

Elements of a Matrix

The elements of a matrix are often denoted by small letters with subscripts. For example:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Where a_{11} = element in row 1, column 1, a_{12} = element in row 1, column 2 etc

Two matrices are equal when they have the same order and corresponding elements are equal.

if $A = \begin{bmatrix} 3 & 9 & -4 \\ 0 & -5 & -2 \\ 1 & 0 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 9 & -4 \\ 0 & -5 & -2 \\ 1 & 0 & -2 \end{bmatrix}$ then $A = B$

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Addition and Subtraction of Matrices

When matrices have the same order they can be added and subtracted by simply adding or subtracting corresponding elements.

Example 2

$$A = \begin{bmatrix} -4 & 0 \\ 3 & 8 \\ -5 & 7 \end{bmatrix} \text{ and } B = \begin{bmatrix} -3 & 7 \\ 8 & -3 \\ 4 & -5 \end{bmatrix},$$

Calculate i) $A - B$ ii) $3A - 2B$

$$\text{i) } A - B = \begin{bmatrix} -4 & 0 \\ 3 & 8 \\ -5 & 7 \end{bmatrix} - \begin{bmatrix} -3 & 7 \\ 8 & -3 \\ 4 & -5 \end{bmatrix} = \begin{bmatrix} -1 & -7 \\ -5 & 11 \\ -9 & 12 \end{bmatrix}$$

$$\text{ii) } 3A - 2B$$

$$3A = 3 \begin{bmatrix} -4 & 0 \\ 3 & 8 \\ -5 & 7 \end{bmatrix} = \begin{bmatrix} -12 & 0 \\ 9 & 24 \\ -15 & 21 \end{bmatrix}$$

$$2B = 2 \begin{bmatrix} -3 & 7 \\ 8 & -3 \\ 4 & -5 \end{bmatrix} = \begin{bmatrix} -6 & 14 \\ 16 & -6 \\ 8 & -10 \end{bmatrix}$$

$$\text{So } 3A - 2B = \begin{bmatrix} -12 & 0 \\ 9 & 24 \\ -15 & 21 \end{bmatrix} - \begin{bmatrix} -6 & 14 \\ 16 & -6 \\ 8 & -10 \end{bmatrix} = \begin{bmatrix} -6 & -14 \\ -7 & 30 \\ -23 & 31 \end{bmatrix}$$

Note that you can easily do the above calculations on the calculator.

Multiplying Matrices

Two matrices can be multiplied if the number of columns in the first matrix is the same as the number of rows in the second matrix.

If matrix A is of order $m \times n$ and matrix B is of order $n \times p$ then the product $A \times B$ exists and will be of order $m \times p$.

Example 3.

$$A = \begin{bmatrix} 3 & 2 \\ -6 & 0 \\ -4 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} -3 & 2 \\ 0 & -6 \end{bmatrix}, \text{ find } A \times B \text{ if it exists.}$$

Since A is a 3×2 matrix and B is a 2×2 matrix the product exists and will be a 3×2 matrix.



Matrix will be 3×2

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$$A \times B = \begin{bmatrix} 3 & 2 \\ -6 & 0 \\ -4 & 4 \end{bmatrix} \times \begin{bmatrix} -3 & 2 \\ 0 & -6 \end{bmatrix} = \begin{bmatrix} 3 \times -3 + 2 \times 0 & 3 \times 2 + 2 \times -6 \\ -6 \times -3 + 0 \times 0 & -6 \times 2 + 0 \times -6 \\ -4 \times -3 + 4 \times 0 & -4 \times 2 + 4 \times -6 \end{bmatrix} = \begin{bmatrix} -9 & -6 \\ 18 & -12 \\ 12 & -32 \end{bmatrix}$$

$$A \times B = \begin{bmatrix} -9 & -6 \\ 18 & -12 \\ 12 & -32 \end{bmatrix} \text{ which is a } 3 \times 2 \text{ matrix.}$$

Note that the matrix product $B \times A$ does **not** exist since.

2×2 and 3×2



unequal so product
does **not** exist

In general $A \times B \neq B \times A$, so the order of matrix multiplication is important!

Matrices and Equations

The Identity Matrix for Multiplication

The identity matrix for multiplication is a square matrix in which all the elements are zero except those in the leading diagonal which are 1.

Examples of identity matrices:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{The identity matrix is always denoted by } I$$

When any matrix is multiplied by the identity matrix that matrix will remain unchanged. It is like the number 1 in normal multiplication. In normal multiplication 1 is the identity for multiplication of all numbers as any number multiplied by 1 is not changed.

For example: $1 \times 21 = 21 = 21 \times 1$ Notice that multiplying 21 by 1 leaves 21 unchanged.

$$\text{If } A = \begin{bmatrix} 1 & 4 \\ 3 & 5 \end{bmatrix} \text{ and } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ then } AI = \begin{bmatrix} 1 & 4 \\ 3 & 5 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 3 & 5 \end{bmatrix}$$

$$\text{and } IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 4 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 3 & 5 \end{bmatrix}$$

Note that $AI = IA$. This is an exception to the rule $AB \neq BA$

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The Inverse of a Matrix

When you multiply 4 by $\frac{1}{4}$ you will get the answer 1. ie. $4 \times \frac{1}{4} = 1 = \frac{1}{4} \times 4$. We say that $\frac{1}{4}$ is the multiplicative inverse of 4. Also 4 is the multiplicative inverse of $\frac{1}{4}$.

Consider the following product:
$$\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} \times \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The result is the identity matrix I . We say that $\begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$ is the multiplicative inverse of $\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ and $\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ is the multiplicative inverse of $\begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$

$$\text{Also } \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

In matrices we use the symbol A^{-1} to denote the multiplicative inverse of A .

So if $A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ then its multiplicative inverse is given by:

$$A^{-1} = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$$

and
$$AA^{-1} = I = A^{-1}A$$

Only **square** matrices have multiplicative inverses.

So in general for any square matrix A :

If $AA^{-1} = I = A^{-1}A$ then A^{-1} is called the multiplicative inverse of A

Finding the Multiplicative Inverse of a 2 by 2 matrix.

In order to find the inverse of a 2 by 2 matrix, you need to find the determinant of the given matrix first.

The Determinant of a 2 by 2 Matrix

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then the determinant of matrix A , denoted by $|A|$ or $\det(A)$, is given by:

$|A| = ad - bc$. Note that if the determinant of a matrix is zero then that matrix is called a **singular matrix**.

Example 1: Find the determinant of $A = \begin{bmatrix} 3 & -5 \\ 2 & -4 \end{bmatrix}$

$$\det(A) = 3 \times (-4) - (-5) \times 2 = -12 - (-10) = -12 + 10 = -2$$

Example 2: Find the determinant of $B = \begin{bmatrix} 4 & 8 \\ 3 & 6 \end{bmatrix}$

$$\det(B) = 4 \times 6 - 8 \times 3 = 24 - 24 = 0 \text{ Matrix B is } \mathbf{singular} \text{ because its determinant is } \mathbf{zero}.$$

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The multiplicative inverse of any 2 by 2 matrix can be found using the following steps, provided the inverse exists.

$$\text{If } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

a. Find the determinant of the matrix.

$$\det(A) = ad - bc.$$

b. Multiply the matrix by $\frac{1}{\det(A)} = \frac{1}{ad-bc}$

$$\frac{1}{ad-bc} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

c. Swap the elements in the main diagonal and change the sign of the elements in the other diagonal. The inverse matrix $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Note that the inverse of a matrix can be easily found using the calculator. Simply raise the matrix to the power of -1.

Example 3: If $A = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$, find the inverse, A^{-1}

First find the determinant of A. $\det(A) = 3 \times 4 - 5 \times 2 = 2$

$$A^{-1} = \frac{1}{2} \begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix}$$

Check that $A \times A^{-1} = I$

$$\begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix} \times \frac{1}{2} \begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix} \times \begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Note that if the matrix is singular (the determinant is zero) then the inverse of the matrix will **not** exist. This is because $\frac{1}{\det(A)} = \frac{1}{0}$ which is not defined. You **cannot** divide by zero!

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Solving Matrix Equations

Inverses enable us to easily solve matrix equations. We use the following technique:

a. If $AX = B$, where A, X and B are matrices. We can find the matrix X by doing the following:

Pre multiply both sides of the equation by A^{-1}

$$A^{-1}AX = A^{-1}B$$

$$IX = A^{-1}B, \quad \text{since } A^{-1}A = I$$

$$X = A^{-1}B \quad \text{since } IX = X$$

In general if $AX = B$ then $X = A^{-1}B$

b. If $XA = B$, where A, X and B are matrices. We can find the matrix X by doing the following:

Post multiply both sides of the equation by A^{-1}

$$XAA^{-1} = BA^{-1}$$

$$XI = BA^{-1}, \quad \text{since } AA^{-1} = I$$

$$X = BA^{-1} \quad \text{since } XI = X$$

In general if $XA = B$ then $X = BA^{-1}$

Example 4

If $A = \begin{bmatrix} 5 & 5 \\ 0 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & -3 \\ 3 & 3 \end{bmatrix}$ find X if

a. $AX = B$ and b. $XA = B$

a. For $AX = B$

$$X = A^{-1}B$$

$$\text{So } X = \begin{bmatrix} 5 & 5 \\ 0 & 6 \end{bmatrix}^{-1} \times \begin{bmatrix} 6 & -3 \\ 3 & 3 \end{bmatrix}$$

The quickest way is to use the **calculator** which will give the following answers, depending on your settings:

$$X = \begin{bmatrix} \frac{7}{10} & -\frac{11}{10} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \text{ or } \begin{bmatrix} 0.7 & -1.1 \\ 0.5 & 0.5 \end{bmatrix} \quad \text{The first answer can be simplified to } \frac{1}{10} \begin{bmatrix} 7 & -11 \\ 5 & 5 \end{bmatrix}$$

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b. For $XA = B$

$$X = BA^{-1}$$

$$X = \begin{bmatrix} 6 & -3 \\ 3 & 3 \end{bmatrix} \times \begin{bmatrix} 5 & 5 \\ 0 & 6 \end{bmatrix}^{-1}$$

The quickest way is to use the **calculator** which will give the following answers, depending on your settings:

$$X = \begin{bmatrix} \frac{6}{5} & -\frac{3}{2} \\ \frac{3}{5} & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 1.2 & -1.5 \\ 0.6 & 0 \end{bmatrix} \quad \text{The first answer can be simplified to } \frac{1}{10} \begin{bmatrix} 12 & -15 \\ 6 & 0 \end{bmatrix}$$

Notice that the answers to part a and part b are different. The order of multiplication is important.

Application of Matrices to Simultaneous Equations

We can use a similar approach to solve a system of simultaneous equations with two unknowns.

Example 5

Solve the following simultaneous equations using matrix methods:

$$2x + 3y = 13$$

$$5x + 2y = 16$$

We can write the system of equations as a matrix equation $AX = B$ as shown below.

$$\begin{bmatrix} 2 & 3 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 13 \\ 16 \end{bmatrix}$$

Notice that A is the matrix of the coefficients, X is a column matrix of the pronumerals x and y and B is a column matrix of the values on the right hand side of the equations.

We can now use $X = A^{-1}B$ to solve the simultaneous equations.

$$\text{ie. } \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 13 \\ 16 \end{bmatrix}$$

$$\text{Since } \begin{bmatrix} 2 & 3 \\ 5 & 2 \end{bmatrix}^{-1} = \frac{1}{4-15} \begin{bmatrix} 2 & -3 \\ -5 & 2 \end{bmatrix} = \frac{1}{-11} \begin{bmatrix} 2 & -3 \\ -5 & 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-11} \begin{bmatrix} 2 & -3 \\ -5 & 2 \end{bmatrix} \times \begin{bmatrix} 13 \\ 16 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-11} \begin{bmatrix} -22 \\ -33 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \text{So } x = 2 \text{ and } y = 3.$$

Or you can use the **calculator** to solve for x and y .

$$\text{Simply enter } \begin{bmatrix} 2 & 3 \\ 5 & 2 \end{bmatrix}^{-1} \times \begin{bmatrix} 13 \\ 16 \end{bmatrix} \text{ to give } \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \text{So } x = 2 \text{ and } y = 3$$

It is worth checking your answers by substituting the values in the original equations.

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Solving Simultaneous Equations with 3 Unknowns.

We can solve a system of simultaneous equations with 3 unknowns using matrices.

Example 6

Solve

$$2x + y + 4z = 17$$

$$3x - 2y = -3$$

$$x + 4y + 5z = 7$$

Write the equations in matrix form $AX = B$

$$\begin{bmatrix} 2 & 1 & 4 \\ 3 & -2 & 0 \\ 1 & 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 17 \\ -3 \\ 7 \end{bmatrix}$$

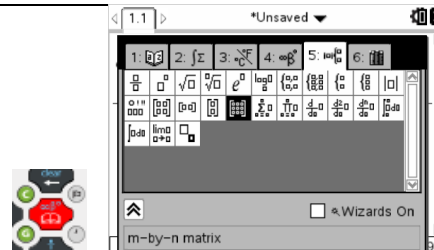
Notice that 0 is inserted for the missing z value in the second equation.

We can now use $X = A^{-1}B$ to solve the simultaneous equations.

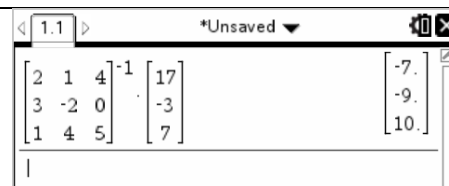
The method for finding the inverse, A^{-1} , of a 3 by 3 matrix manually is complex and is not in the course. **You will need to use the calculator.**

$$\text{So } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & 1 & 4 \\ 3 & -2 & 0 \\ 1 & 4 & 5 \end{bmatrix}^{-1} \times \begin{bmatrix} 17 \\ -3 \\ 7 \end{bmatrix}$$

Step 1: In calculator view, hit the Catalogue button and choose Tab 5. Hit the 3 by 3 matrix template



Step 2: Fill the 3 by 3 template with the appropriate values and raise the matrix to the power of -1 by using the button \wedge . Press the multiply button and choose the 3 by 3 template again and adjust the number of columns to 1. Press Enter.



$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -7 \\ -9 \\ 10 \end{bmatrix} \quad \text{So } x = -7, y = -9, z = 10$$

Check the answers by substituting the values in the original equations.

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Sometimes simultaneous equations **cannot** be solved. This occurs when the **determinant is zero and the inverse does not exist**. We say that the equations do not have a unique solution.

Example 7

Solve

$$3x + 2y = 9$$

$$6x + 4y = 22$$

We can write the system of equations as a matrix equation $AX = B$ as shown below.

$$\begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9 \\ 22 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}^{-1} \times \begin{bmatrix} 9 \\ 22 \end{bmatrix}$$
 However the inverse does not exist, because the determinant of $\begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}$ is 0.

There is no solution to the equations. This is because they are parallel lines and do not meet.

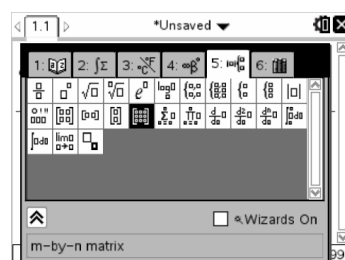
The Determinant of a 3 by 3 Matrix

The calculation of the determinant of a 3 X 3 matrix manually is not in the course. It can easily be found **using a calculator**.

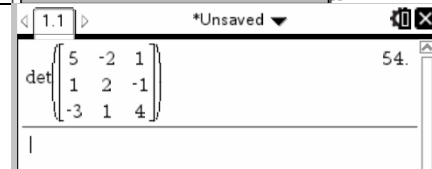
Example 8

If $A = \begin{bmatrix} 5 & -2 & 1 \\ 1 & 2 & -1 \\ -3 & 1 & 4 \end{bmatrix}$, find the $\det(A)$

Step 1: In calculator view, hit Menu, Matrix and Vector, Determinant.
Hit the Catalogue button and choose Tab 5. Hit the 3 by 3 matrix template



Step 2:
Enter the values.
Hit the right arrow button to ensure the pointer is to the right of the matrix and press Enter.



The determinant is 54.

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Transition Matrices

Matrices can be used in probabilities to model situations where there is a transition from one state to the next. The next state's probability is conditional on the result of the preceding outcome.

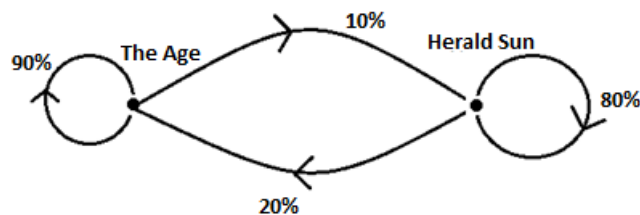
Example

In Melbourne there are two major daily newspapers, The Age and the Herald Sun. Readers are fairly loyal to the newspaper they intend to buy. Records in a particular country town have shown that of the people who purchase a newspaper every day, 90% of people who buy The Age on one day will buy it the next day and 80% of people who buy the Herald Sun on one day will buy it the next day.

Draw up a table that can be used to form a transition matrix.

	Buying <i>The Age</i> on the first day	Buying the <i>Herald-Sun</i> on the first day
Buying <i>The Age</i> on the next day	0.9	0.2
Buying the <i>Herald-Sun</i> on the next day	0.1	0.8

We can represent this information as a transition diagram.



or the information can be represented in a transition matrix T

$$T = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix}$$

The elements contained in transition matrices represent conditional probabilities, each of which tells us the probability of an event occurring given that another event has previously occurred. In this case it is the probability of buying a particular newspaper on the next day given that a particular newspaper had been purchased on the previous day.

Notice that in a transition matrix the sum of the probabilities in each column is 1. This can be used as a quick check to ensure that the probabilities have been entered correctly.

Initial State Matrix denoted by s_0

In the example above suppose that on a certain day 600 copies of The Age are sold and 1000 copies of the Herald Sun are sold. Use this information to predict the number of copies of each paper that will be sold on the next three days.

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The initial state matrix, s_0 , is a 2 by 1 column matrix denoted by:

$$s_0 = \begin{bmatrix} 600 \\ 1000 \end{bmatrix} \quad \begin{array}{l} \textit{The Age} \\ \textit{Herald Sun} \end{array}$$

We can form the state matrix, s_1 which gives the state on the next day. That is the predicted number of people who buy The Age newspaper and the Herald Sun newspaper the next day.

$$\begin{aligned} s_1 = Ts_0 &= \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \begin{bmatrix} 600 \\ 1000 \end{bmatrix} \\ &= \begin{bmatrix} 0.9 \times 600 + 0.2 \times 1000 \\ 0.1 \times 600 + 0.8 \times 1000 \end{bmatrix} \\ &= \begin{bmatrix} 540 + 200 \\ 60 + 800 \end{bmatrix} \\ &= \begin{bmatrix} 740 \\ 860 \end{bmatrix} \end{aligned}$$

On the next day it is predicted that 740 copies of The Age will be sold and 860 copies of the Herald Sun will be sold. Notice that $740 + 860$ will total 1600. This agrees with the initial total of newspapers sold.

The number of newspapers sold on the **second** day is can be predicted to be:

$$\begin{aligned} s_2 = Ts_1 &= \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \begin{bmatrix} 740 \\ 860 \end{bmatrix} \\ &= \begin{bmatrix} 0.9 \times 740 + 0.2 \times 860 \\ 0.1 \times 740 + 0.8 \times 860 \end{bmatrix} \\ &= \begin{bmatrix} 666 + 172 \\ 74 + 688 \end{bmatrix} \\ &= \begin{bmatrix} 838 \\ 762 \end{bmatrix} \end{aligned}$$

Again check that $838 + 762 = 1600$

On the second day it is predicted that 838 copies of The Age will be sold and 762 copies of the Herald Sun will be sold.

Similarly on the third day the predicted number of each type of newspaper sold is given by:

$$s_3 = Ts_2 = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \begin{bmatrix} 838 \\ 762 \end{bmatrix} = \begin{bmatrix} 906.6 \\ 693.4 \end{bmatrix}$$

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On the third day it is predicted that 907 copies of The Age will be sold and 693 copies of the Herald Sun will be sold. Again check that $906.6 + 693.4 = 1600$

Similarly for the fourth day the predicted number of newspapers sold is given by:

$$s_4 = Ts_3$$

$$s_4 = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \begin{bmatrix} 906.6 \\ 693.4 \end{bmatrix} = \begin{bmatrix} 954.62 \\ 645.38 \end{bmatrix}$$

In general the next state matrix can be calculated in terms of the previous state matrix by using the formula:

$$s_n = Ts_{n-1}$$

There is an alternative formula for finding the predicted number of each type of newspaper sold on a particular day. Suppose we need to find s_4 , the number of each type of newspaper sold on the fourth day.

A more direct way of finding s_4 is to calculate T^4s_0

$$\text{Using the calculator } T^4s_0 = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix}^4 \begin{bmatrix} 600 \\ 1000 \end{bmatrix} = \begin{bmatrix} 954.62 \\ 645.38 \end{bmatrix} \quad (\text{This gives same result as above})$$

On the fourth day 955 copies of The Age will be sold and 645 copies of the Herald Sun will be sold.

$$\text{In general } s_n = T^n s_0$$

This formula is very important as it gives a direct way of calculating the n_{th} state using the product of the n_{th} power of the transition matrix and the initial state matrix. You will need to use the calculator to find s_n . **Remember to raise the transition matrix to the required power!**

To calculate the number of each type of newspaper sold on the 10th day use:

$$s_{10} = T^{10}s_0 = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix}^{10} \begin{bmatrix} 600 \\ 1000 \end{bmatrix} = \begin{bmatrix} 1053.48 \\ 546.516 \end{bmatrix}$$

On the 10th day 1053 copies of The Age will be sold and 547 copies of the Herald Sun will be sold.

(Note that if we use the other formula $s_{10} = Ts_9$, we would need to calculate $s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8$ and s_9 first. This would be extremely time consuming!)

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Steady State

Often the values of a state matrix stabilise as n increases. One way to check that in the long term the state matrix remains steady is to test a large value of n such as 50 and then test the next value of n 51. If the elements in the matrix have not changed then a steady state has been reached.

When there is no noticeable change from one state matrix to the next, the system is said to have reached its steady state.

In the previous example let's test for steady state by finding the number of each type of newspaper sold on the 50th and the 51st day:

$$T^{50}S_0 = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix}^{50} \begin{bmatrix} 600 \\ 1000 \end{bmatrix} = \begin{bmatrix} 1066.67 \\ 533.333 \end{bmatrix}$$


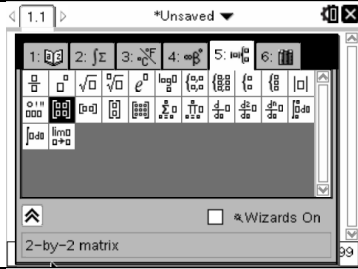
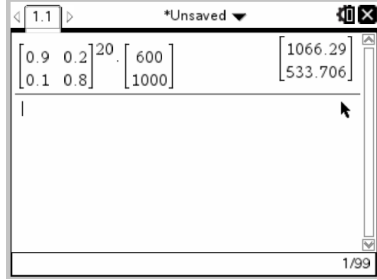
$$T^{51}S_0 = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix}^{51} \begin{bmatrix} 600 \\ 1000 \end{bmatrix} = \begin{bmatrix} 1066.67 \\ 533.333 \end{bmatrix}$$

Since the number of newspapers is the same for day 50 and day 51, the steady state has been reached.

Time to reach steady state.

Suppose you need to find the minimum number of days for the number of newspapers to reach the steady state values given above.

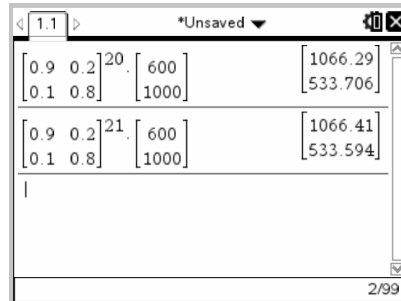
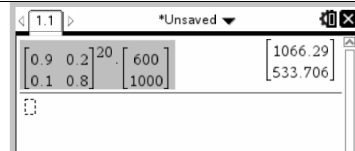
This is not so straight forward. You will need to use your calculator testing out increasing powers in the transition matrix until the steady state values appear on the calculator. You need to learn to use the calculator efficiently for this type of question. The following steps give one approach.

<p>Step 1: In Calculator View, hit the catalogue button.</p>	
<p>Step 2: Ensure tab 5 is selected and select the 2by 2 matrix template.</p>	
<p>Step 3: Fill in the 2 by 2 transition matrix and raise it to the power 20 by using the ^ button on the calculator. Hit the multiply button and using the catalogue button enter the column matrix as shown opposite. Press Enter to perform the multiplication. Notice that the resultant values are slightly different from the steady state values found earlier.</p>	

Further Maths Matrix Summary

Step 4: Copy the matrix calculation. This is easily done by pressing the up arrow button twice so that the whole expression is highlighted and then pressing the Enter key.

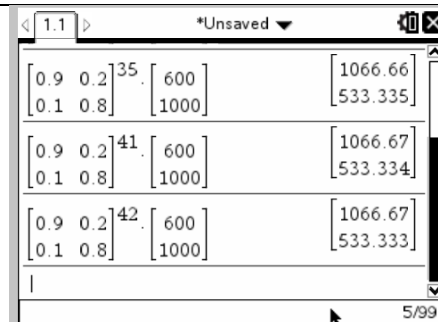
You can now easily edit the matrix expression by changing the power to 21. Press Enter to perform the calculation.



Step 5: Repeat this process until the steady state numbers are reached. That is the result:

$$\begin{bmatrix} 1066.67 \\ 533.333 \end{bmatrix}$$

You will need to test larger powers as shown.



It appears that the steady state is reached after 42 days.